⟨Lecture 26⟩

Representation theory: Let G be a group, V a vector space $/k$.

A representation of G is a hom $\rho: G \to \mathrm{Aut}_{k\text{-vec}}(V) \cong GL_n k$. (f.d.)

If ρ is injective ("faithful") then it lets us think of G as a subgroup of $GL_n k$. So it gives matrices that "represent" the elements.

A rep ρ also makes V into a G-space: $g \cdot v = \rho(g)(v)$.
So a rep is equiv to a G-module structure on V.

Often we study reps where G, ρ are required to have extra properties.

e.g. G Lie grp ρ smooth $k = \mathbb{R}, \mathbb{C}$
G alg grp ρ regular

There are particularly nice "full" rep theories in cases:

① G finite (Serre's book)
② G compact Lie group, ρ smooth, $k = \mathbb{C}$
③ G semisimple linear alg $/\mathbb{C}$, ρ regular, $k = \mathbb{C}$.

Moreover (amazingly), the map $(G, \rho) \longmapsto (K, \rho|_K)$
where K is a maximal cpt subgroup gives an equivalence between classifications ③ and ②.

So I'll talk about case ③. I'll also assume G simply conn.

Fix $H \subset B \subset G$ with assoc root data Φ, Φ^+, Δ.

Recall $\Phi \subset V = \mathfrak{a}^* = \mathbb{R}\text{-span of } \Phi \subset \mathfrak{h}^*$.

$\lambda \in \mathfrak{h}^*$ is called algebraically integral if $2\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \ \forall \alpha \in \Phi$

Thus for example Φ is alg int. Alg int $\Rightarrow \lambda \in \mathfrak{a}^*$.

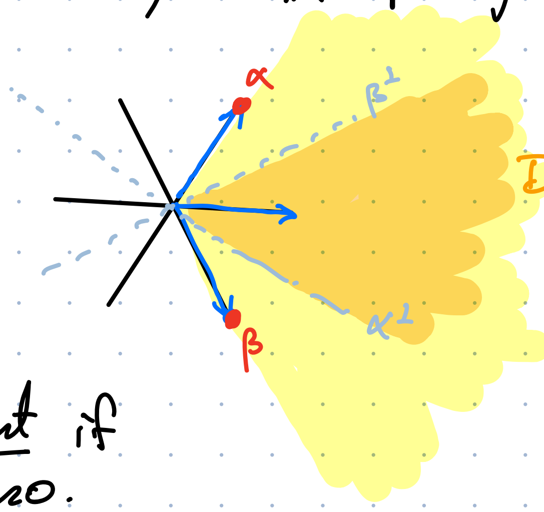
Two ways to compare elements of $\mathfrak{a}^*$: Recall $\langle , \rangle$ W-inv't inner prod.

λ dominates μ: $\langle \lambda, \alpha \rangle \geq \langle \mu, \alpha \rangle$ for all $\alpha \in \Phi^+$ (or $\langle \lambda - \mu, \alpha \rangle \geq 0$)

λ higher than μ: $\lambda - \mu \in \mathbb{R}^+$-span of Φ^+.

e.g. for $SL_3\mathbb{C}$:

Φ Φ^+ Δ



λ is called <u>dominant</u> if λ dominates zero.

Let V be a rep of G (ρ implicit). V is irreducible if the only G-inv't subspaces of V are $\{0\}$ and V. V is decomposable if $\exists$ nontrivial G-inv't subsp $W_1, W_2 \subset V$ s.t. $V \cong W_1 \oplus W_2$.

<u>Thm.</u> For semisimple G, V irred $\Leftrightarrow$ V indecomposable.
Thus every f.d. rep is expressible as $W_1 \oplus \cdots \oplus W_k$ w/ W_i irred reps.

<u>Classifying irred reps.</u>

As with $G \curvearrowright \mathfrak{g}$, the vec space V decomposes into weight spaces:

$$V = \bigoplus_{\lambda \in \Gamma_V} V_\lambda \quad \text{where} \quad \exp(t)\cdot x = e^{\lambda(t)} x \quad \forall\, x \in V_\lambda.$$

Here $\Gamma_V \subset \mathfrak{h}^*$ are the <u>weights</u> of the rep.

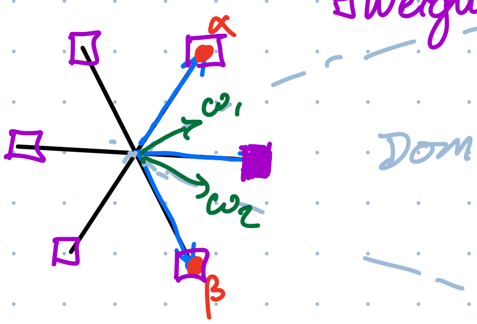
G simply conn $\Rightarrow$ (alg int $\Leftrightarrow$ analytically int)

Theorem of the highest weight: Γ_V contains a unique element higher than all the other weights. This highest weight is dominant, integral, and it uniquely determines the rep V up to isomorphism. Furthermore, every dominant int weight arises.

The last point means $\forall\ \lambda$ dominant integral, there's some vec space $V(\lambda)$ and $\rho: G \to V(\lambda)$.

Where does $V(\lambda)$ come from?

□ Weights of Ad. ■ highest weight of Ad $= \alpha+\beta$



Dominant

Dominant + integral $= \mathbb{Z}^{\geq 0}$-span ω_1, ω_2

Hint of constr: B has a fixed pt on $\mathbb{P}(V)$. Ends up being unique and giving vector v in V_λ, λ highest.

Recall $B \curvearrowright \mathbb{C}_\lambda$ means $b \cdot z = \tilde{\lambda}(b)^{-1} z$

And $\mathcal{L}_\lambda = (G \times \mathbb{C}_\lambda)/B$ line bdl over G/B

Def. $H^0(G/B, \mathcal{L}_\lambda) = \{$ holo B-equivt maps $G \to \mathbb{C}_\lambda \}$
$=$ holo maps $G/B \to \mathcal{L}_\lambda$ inverse to proj.

Compactness of G/B can be used to show this space is findim.

G acts: $g \cdot f(a) = f(g \cdot a)$.

So $H^0(G/B, \mathcal{L}_\lambda)$ is a G representation. Note λ already integ.

<u>Theorem</u> (Borel-Weil) If λ is dominant, then $H^0(G/B, \mathcal{L}_\lambda)$ is G-equivt iso to $V(\lambda)^*$. Otherwise, $H^0(G/B, \mathcal{L}_\lambda) = \{0\}$.

<u>Sketch</u> Start w/ polynomial fns $G \to \mathbb{C}$. ($\mathbb{C}[G]$) Define right B-action

$$(f.b)(g) = \overset{v.s}{\lambda}(b) f(gb). \qquad \text{(This is } \mathbb{C}[G] \otimes \mathbb{C}_\lambda)$$

Then $H^0(G/B, \mathcal{L}_\lambda) = (\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^B$

Peter-Weyl thm: $\mathbb{C}[G] \cong \bigoplus_{\substack{\mu \text{ dominant} \\ \text{integral}}} V(\mu)^* \otimes V(\mu)$

$$(\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^B \cong \bigoplus_\mu \left(V(\mu)^* \otimes V(\mu) \otimes \mathbb{C}_\lambda\right)^B.$$

$V(\mu)$ has a unique B-invt line, iso to $\mathbb{C}_{-\mu}$

$$(\mathbb{C}[G] \otimes \mathbb{C}_\lambda)^B \cong \bigoplus_\mu V(\mu)^* \otimes \underbrace{(\mathbb{C}_{-\mu} \otimes \mathbb{C}_\lambda)^H}_{\substack{\mathbb{C} \text{ if } \mu = \lambda \\ 0 \text{ else.}}} = V(\lambda)^*$$

$\square$